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Math 451: Intro. to General Topology

Homework 1

Due: Jan 21, 23:59

1. Let A, B be sets and $f : A \to B$ a function. Let $A' \subseteq A$ and $B' \subseteq B$.

(a) Prove that $f^{-1}\big(f(A')\big) \supseteq A'$. Find a sufficient condition for the equality to hold and prove that it is indeed sufficient.

(b) Prove that $f\big(f^{-1}(B')\big) \subseteq B'$. Find a sufficient condition for the equality to hold and prove that it is indeed sufficient.

2. Let $f : A \to B$ be a function and let $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$.

(a) Prove that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(b) Prove that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(c) Prove that $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$.

(d) Provide an example such that $f(A_1 \setminus A_2) \neq f(A_1) \setminus f(A_2)$. Provide a condition on f that implies that $f(A_1 \setminus A_2) = f(A_1) \setminus f(A_2)$.

3. (a) **Disjointification.** Let $\mathcal{S}$ be a set of sets (which may overlap). For a set S, let $\widetilde{S} := \{(x, S) : x \in S\}$ and let $\widetilde{\mathcal{S}} := \left\{\widetilde{S} : S \in \mathcal{S}\right\}$. Show that any two distinct sets $\widetilde{S}_0, \widetilde{S}_1$ in $\widetilde{\mathcal{S}}$ are disjoint (i.e. have empty intersection).

(b) Use the previous part to prove that Axiom of Choice follows from the statement "Every surjection has a right inverse."

Hint: Given a set $\mathcal{S}$ of nonempty sets, define a surjection $f : \bigcup \widetilde{\mathcal{S}} \twoheadrightarrow \mathcal{S}$.

4. Prove without Axiom of Choice that if a set X is Dedekind infinite (i.e. there is an injection $\iota : X \hookrightarrow X$ which is not surjective), then $\mathbb{N} \hookrightarrow X$.

Hint: Let x_0 be any element of $X \setminus f(X)$. Show that $\iota^n(x_0) \neq \iota^m(x_0)$ for distinct $n, m \in \mathbb{N}$.

5. Prove that for sets X, Y, if X is infinite and Y is countable then $X \cup Y \equiv X$.

Hint: Show that there is a sequence $(x_n)_{n \in \mathbb{N}}$ of pairwise distinct elements and prove $\{x_n\} \equiv \{x_n\} \cup Y$.

6. Prove directly, without using $\mathscr{P}(\mathbb{N}) \equiv [0,1]$, that $[0,1]$ is uncountable.

Hint: Write reals in $[0,1]$ in their decimal expansion $0.d_0 d_1 d_2 \ldots$ and apply Cantor's diagonalization method an enumeration of all reals in $[0,1]$.

7. Let $n \in \mathbb{N}^+$ (positive natural numbers). Define the p-**norm** $\|x\|_p$ for all $p \geqslant 1$ and the ∞-norm $\|x\|_\infty$ of a vector $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ by

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$
$$\|x\|_\infty := \max_i |x_i|.$$

Note that for all $1 \leqslant p \leqslant \infty$, we have $\|\alpha x\|_p = |\alpha| \|x\|_p$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

(a) Fill in all the details in the outline below of a proof of Minkovski's inequality:

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p$$

for all $x, y \in \mathbb{R}^n$ for all $1 \leqslant p \leqslant \infty$.

Outline: The proof for $p = \infty$ is straightforward. For $p \in [1, \infty)$, we may assume that $x \neq 0$ and $y \neq 0$, so $\|x\|_p + \|y\|_p > 0$. Dividing both sides by $\|x\|_p + \|y\|_p$ and raising to the p^{th} power, it is enough to prove that

$$\|tx' + (1 - t)y'\|_p^p \leqslant 1,$$

where $x' := \frac{x}{\|x\|_p}$, $y' := \frac{y}{\|y\|_p}$, and $t := \frac{\|x\|_p}{\|x\|_p + \|y\|_p}$. But $\|tx' + (1-t)y'\|_p^p \leqslant \sum_{i=1}^n (t|x_i'| + (1-t)|y_i'|)^p$ and you may use without proof that the function $\beta \mapsto \beta^p$ is convex. i.e. $(t|x_i'| + (1 - t)|y_i'|)^p \leqslant t|x_i'|^p + (1 - t)|y_i'|^p$. Finally, one observes that $t|x_i'|^p + (1 - t)|y_i'|^p \leqslant 1$.

(b) Deduce that $d_p(x, y) := \|x - y\|_p$ is a metric on $\mathbb{R}^n$.

8. Let $n \in \mathbb{N}^+$.

(a) Prove that $\|x\|_\infty \leqslant \|x\|_p \leqslant n^{1/p} \|x\|_\infty$ for all $x \in \mathbb{R}^n$ and all $p \in [1, \infty]$.

(b) Deduce that the metrics d_p are all Lipschitz equivalent to each other for $p \in [1, \infty]$.

(c) Also deduce that $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$ for all $x \in \mathbb{R}^n$.